Maximal Subgroups of the Coxeter Group $W(H_4)$ and Quaternions

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The largest finite subgroup of $O(4)$ is the noncrystallographic Coxeter group $W(H_4)$ of order 14400. Its derived subgroup is the largest finite subgroup $W(H_4)/Z_2$ of $SO(4)$ of order 7200. Moreover, up to conjugacy, it has five non-normal maximal subgroups of orders 144, two 240, 400 and 576. Two groups $[W(H_2)\times W(H_2)]\times Z_4$ and $W(H_3)\times Z_2$ possess noncrystallographic structures with orders 400 and 240 respectively. The groups of orders 144, 240 and 576 are the extensions of the Weyl groups of the root systems of $SU(3)\times SU(3)$, $SU(5)$ and $SO(8)$ respectively. We represent the maximal subgroups of $W(H_4)$ with sets of quaternion pairs acting on the quaternionic root systems.

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INTRODUCTION

The noncrystallographic Coxeter group $W(H_4)$ of order 14400 generates some interests [1] for its relevance to the quasicrystallographic structures in condensed matter physics [2, 3] as well as its unique relation with the E_8 gauge symmetry associated with the heterotic superstring theory [4]. The Coxeter group $W(H_4)$ [5] is the maximal finite subgroup of $O(4)$, the finite subgroups of which have been classified by du Val [6] and by Conway and Smith [7]. It is also one of the maximal subgroups of the Weyl group $W(E_8)$ splitting 240 nonzero roots of E_8 into two equal size disjoint sets. One set can be represented by the icosians q (quaternionic elements of the binary icosahedral group I) and the remaining set is σq [8, 9] where $\sigma = \frac{1-\sqrt{5}}{2}$. Embedding of $W(H_4)$ in $W(E_8)$ is studied in detail in references [10] and [11].

In this paper we study the maximal subgroups of $W(H_4)$ and show that it possesses, up to conjugacy, five maximal subgroups of orders 144, 240, 400 and 576. They correspond to the symmetries of certain Dynkin and Coxeter diagrams. Another obvious maximal subgroup $W(H_4)' \equiv W(H_4)/Z_2$ of $W(H_4)$ of order 7200 is also the largest finite subgroup of $SO(4)$. Two groups of orders 400 and 240 are related to the symmetries of the noncrystallographic Coxeter graphs $H_2 \oplus H_2'$ and H_3 respectively. The remaining groups of orders 144, 240 and 576 are associated with the symmetries of the crystallographic root systems $A_2 \oplus A_2'$, A_4 and D_4 respectively (we interchangeably use $A_2 \approx SU(3), A_4 \approx SU(5), D_4 \approx SO(8)$). Embeddings of these groups in $W(H_4)$ are not trivial and the main objective of this work is to clarify this issue.

In the context of quaternionic representation of the H_4 root system by the elements of binary icosahedral group, identifications of the quaternionic root systems of the maximal subgroups will be simple. The maximal subgroups of orders 144, 400 and 576 associated with the root systems $A_2 \oplus A_2'$, $H_2 \oplus H_2'$ and D_4 respectively have close relations with the maximal subgroups of the binary icosahedral group I since it has three maximal subgroups. However the maximal subgroups associated with the Coxeter diagram H_3 and the Dynkin diagram A_4 do not have such correspondences and some care should be undertaken.

In Sec. we introduce the root system of H_4 in terms of the icosions I and identify the maximal subgroups of the binary icosahedral group. We discuss briefly the method as to how the group elements of $W(H_4)$ are obtained from icosians. In Sec. , we study in terms of quaternions, the group structure associated with the graph $A_2 \oplus A_2'$. Sec. is devoted to a similar analysis of the symmetries of the noncrystallographic root system $H_2 \oplus H_2'$. In Sec. we deal with the extension of $W(D_4)$ with a cyclic symmetry of its Dynkin diagram, the group of order 576, which was also studied in a paper of ours [12] in a different context. Sec. is devoted to the study of the automorphism of the H_3 root system and we show that it is, up to conjugacy, a group of order 240 when it acts in the 4-dimensional space. In Sec. we study the somewhat intricate structure of the automorphism group of A_4 using quaternions.

H_4 WITH ICOSIANS

The root system of the Coxeter diagram H_4 consists of 120 roots which can be represented by the quaternionic elements of the binary icosahedral group I given in Table I (so is called the group of icosians). They can be generated by reflections on the simple roots depicted in Fig 1.

Any real quaternion can be written as $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ where q_a $(a = 0, 1, 2, 3)$ are real numbers and pure quaternion units[14] e_i $(i = 1, 2, 3)$ satisfy the well-known relations

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \quad (i, j, k = 1, 2, 3) \tag{1}$$

where ϵ_{ijk} is the Levi-Civita symbol. The scalar product of two quaternions p and q is defined by

$$(p, q) = \frac{1}{2}(p\overline{q} + q\overline{p}) \tag{2}$$

which leads to the norm $N(q) = (q, q) = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

TABLE I: Conjugacy classes of the binary icosahedral group I represented by quaternions.(Here $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$)

Conjugacy Classes and orders of elements	Elements of the conjugacy classes also denoted by their numbers (Cyclic permutations in e_1, e_2, e_3 should be added if not included)
1	1
2	-1
10	$12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3)$
5	$12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3)$
10	$12'_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2)$
5	$12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2)$
6	$20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(1 \pm \tau e_1 \pm \sigma e_2)$
3	$20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(-1 \pm \tau e_1 \pm \sigma e_2)$
4	$30 :$ $15_+ : e_1, e_2, e_3, \frac{1}{2}(\sigma e_1 \pm \tau e_2 \pm e_3)$ $15_- : -e_1, -e_2, -e_3, \frac{1}{2}(-\sigma e_1 \pm \tau e_2 \pm e_3)$

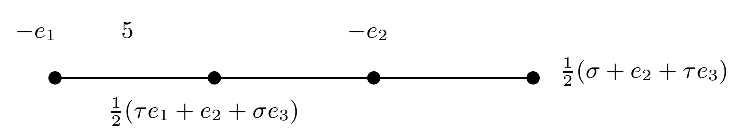

FIG. 1: The Coxeter diagram of H_4 with quaternionic simple roots

The icosians of the binary icosahedral group I are classified in Table I according to the conjugacy classes. Denote by p, q and r any three elements of I. The transformations defined by

$$[p, r] \ : \ q \rightarrow pqr \tag{3}$$
$$[p, r]^* \ : \ q \rightarrow p\overline{q}r \tag{4}$$

preserve the norm $q\overline{q} = \overline{q}q$ as well as leave the set of icosians I intact so that the pairs $[p, r]$ and $[p, r]^*$ represent the group elements of $W(H_4)$. Since $[p, r] = [-p, -r]$ and $[p, -r] = [-p, r]$ the $W(H_4)$ consists of $120 \times 120 = 14400$ elements. Details can be found in reference [11]. The element $[1, 1]^*$ acts as a conjugation, $[1, 1]^* : q \rightarrow \overline{q}$, which is the normalizer of the subgroup $SO(4)$ so that $O(4)$ can be written as the semi-direct product $O(4) \approx SO(4) \times Z_2$. It follows from this structure that one of the maximal subgroup $W(H_4)'$ of order 7200 is obviously a maximal finite subgroup of $SO(4)$. The group $W(H_4)'$ can be represented by the pair $[p, r]$, $p, r \in I$, possessing 42 conjugacy classes.

It is easily found from the structure of the conjugacy classes of I in Table I that the maximal subgroups of I can be generated by the sets of quaternions[15]

$$\text{Dihedral group of order 12} \quad : \quad \frac{1}{2}(1+\tau e_1+\sigma e_2), e_3 \tag{5}$$

$$\text{Dihedral group of order 20} \quad : \quad \frac{1}{2}(\tau+\sigma e_1+e_2), e_3 \tag{6}$$

$$\text{Binary tetrahedral group of order 24} \quad : \quad \frac{1}{2}(1+e_1+e_2+e_3), e_3 \tag{7}$$

These generators are the representative elements of the conjugacy classes of the maximal subgroups of I. The binary icosahedral group has two quaternionic irreducible representations. If we denote by I' the other quaternionic representation it can be obtained from I by interchanging $\sigma \leftrightarrow \tau$ in Table I. We will use both representations in section .

THE MAXIMAL SUBGROUP OF ORDER 144

A little exercise shows that the set of elements in Eq. 5 constitute the root system of the Lie algebra $A_2 \oplus A_2'$ where the simple roots are given in Figure 2.

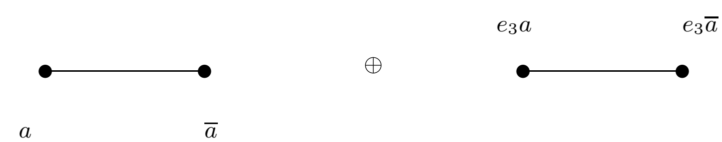

FIG. 2: Dynkin diagram of $A_2 \oplus A_2'$with the elements of dihedral group of order 12 ($a = \frac{1}{2}(1+\tau e_1+\sigma e_2)$, $\overline{a} = \frac{1}{2}(1-\tau e_1-\sigma e_2)$).

We have used one of the four dimensional irreducible matrix representations of $W(H_4)$ to identify its maximal subgroups via computer calculations which resulted in those six maximal subgroups. Our main objective here is to construct the elements of the maximal subgroups using pairs of quaternions described in (3-4). In each section we give arguments - though not very rigorous - that the groups of concern are maximal in $W(H_4)$. We take the computer calculations as evidence for the completeness of the maximal subgroups. The rigorous proof is beyond the scope of this paper.

It follows that the sets $(\pm a, \pm\overline{a}, \pm 1)$, $(a^6 = 1)$ and $(\pm e_3 a, \pm e_3\overline{a}, \pm e_3)$ constitute the roots of A_2 and A_2' respectively. Then the generators of the Weyl groups $W(A_2)$ and $W(A_2')$ are the group elements[16]

$$r_1 = [a, -a]^*, \quad r_2 = [\overline{a}, -\overline{a}]^* \tag{8}$$

$$r_1' = [e_3 a, -e_3 a]^*, \quad r_2' = [e_3\overline{a}, -e_3\overline{a}]^*. \tag{9}$$

The sets of elements of $W(A_2)$ and $W(A_2')$ can be easily generated by (8) and (9):

$$W(A_2) \approx D_3 \approx S_3 : [a, -a]^*, [\overline{a}, -\overline{a}]^*, [1, -1]^*, [a, a], [\overline{a}, \overline{a}], [1, 1] \tag{10}$$

$$W(A_2') \approx D_3 \approx S_3 : [e_3 a, -e_3 a]^*, [e_3\overline{a}, -e_3\overline{a}]^*, [e_3, -e_3]^*, [\overline{a}, a], [a, \overline{a}], [1, 1]. \tag{11}$$

Note that the longest elements of the respective groups are $w_0 = [-1, 1]^*$ and $w_0' = [e_3, -e_3]^*$. The automorphism groups $Aut(A_2)$ and $Aut(A_2')$ of the respective root systems are the extensions of the Weyl groups by the respective Dynkin diagram symmetries. The diagram symmetries of A_2 and A_2' are obtained by the exchange of simple roots $a \leftrightarrow \overline{a}$ and $e_3 a \leftrightarrow e_3\overline{a}$ which respectively lead to the groups

$$Aut(A_2) \approx W(A_2) \rtimes \gamma, \quad Aut(A_2') \approx W(A_2)' \rtimes \gamma'. \tag{12}$$

where $\gamma = [1,1]^*$ and $\gamma' = [e_3, e_3]^*$. The groups in (12) having equal orders 12 commute with each other. It is interesting to observe that the automorphism group $Aut(A_2)$ can be written as a sum of two cosets in two different ways,

$$Aut(A_2) = \{W(A_2), \gamma W(A_2)\} \tag{13}$$

and

$$Aut(A_2) = \{W(A_2), w_0 \gamma W(A_2)\} \tag{14}$$

where $w_0\gamma = c = [-1,1],\ c^2 = [1,1]$ which commutes with $W(A_2)$. Therefore the group $Aut(A_2)$ can also be written as $Aut(A_2) \approx W(A_2) \times Z_2$, where Z_2 is generated by the element c. A similar analysis is true for the diagram A'_2 where $w'_0\gamma' = c$. The direct product of the two Weyl groups and the group Z_2 is of order 72. The elements of the cyclic group $Z_4 =< [e_3, 1] >= \{[\pm 1, 1], [\pm e_3, 1]\}$ transform the two groups $W(A_2)$ and $W(A'_2)$ to each other by conjugation where $[e_3, 1]^2 = c$. Since the group $W(A_2) \times W(A'_2)$ is invariant under Z_4 by conjugation and the only common element to both groups is the unit element $[1, 1]$ then one can extend $W(A_2) \times W(A'_2)$ by Z_4 to $[W(A_2) \times W(A_2)] \rtimes Z_4$. The group $[W(A_2) \times W(A'_2)] \rtimes Z_4$, of four cosets $[1, 1][W(A_2) \times W(A'_2)], [-1, 1][W(A_2) \times W(A'_2)], [e_3, 1][W(A_2) \times W(A'_2)]$, and $[-e_3, 1][W(A_2) \times W(A'_2)]$ has order 144. Moreover it contains $W(A_2) \times W(A'_2) \times Z_2$ as a maximal subgroup of index 2. The element $[e_3, 1]$ acts by exchanging the reflections $r_1$and r'_2 and r_2 and r'_1. This is a diagram automorphism of the root system of $A_2 \oplus A'_2$. Since the Weyl group is extended by the cyclic group Z_4 generated by the element $[e_3, 1]$ the extended group is nothing but $[W(A_2) \times W(A'_2)] \rtimes Z_4 \approx Aut[A_2 \oplus A'_2]$

In terms of the quaternion pairs in (3, 4) the above group can be written compactly

$$[p, r], [p, r]^* \quad \text{with} \quad p, r \in \{\pm a, \pm\overline{a}, \pm 1, \pm e_3 a, \pm e_3\overline{a}, \pm e_3\}. \tag{15}$$

To prove that the group $[W(A_2) \times W(A'_2)] \rtimes Z_4$ is maximal in $W(H_4)$ we argue as follows. Let us denote by $[b, 1]$ an element of $W(H_4)$ not belonging to $[W(A_2) \times W(A'_2)] \rtimes Z_4$, i.e. b is not one of those quaternions in the set of roots $A_2 \oplus A'_2$ in (15). The products of $[b, 1]$ with the group elements $[p, r]$ and $[p, r]^*$ of (15) would yield the elements $[pb, r], [bp, r], [bp, r]^*$ and $[p, \overline{b}r]^*$. These elements obviously do not belong to the group $[W(A_2) \times W(A_2)] \rtimes Z_4$. The elements pb and $\overline{b}r$ and those elements obtained by repetitive multiplications will generate the binary icosahedral group I not any subgroups of it, since the dihedral group of order 12 is already a maximal subgroup of I. Therefore the group generated in this manner will be the whole group $W(H_4) = \{[I, I], [I, I]^*\}$ and the group $[W(A_2) \times W(A'_2)] \rtimes Z_4$ is, up to conjugacy, a maximal subgroup of $W(H_4)$.

Similar arguments apply for the maximality of the groups $[W(H_2) \times W(H'_2)] \rtimes Z_4$ and $W(D_4) \rtimes Z_3$ as they are generated from the maximal dihedral subgroup of order 20 and the maximal binary tetrahedral subgroup of the binary icosahedral group I.

THE SYMMETRY GROUP OF $H_2 \oplus H'_2$ OF ORDER 400

A similar analysis to the one studied in Sec. can be pursued by introducing the root system of the noncrystallographic Coxeter diagram $H_2 \oplus H'_2$. The Coxeter diagram of the system with quaternions is shown in Figure 3. If we denote

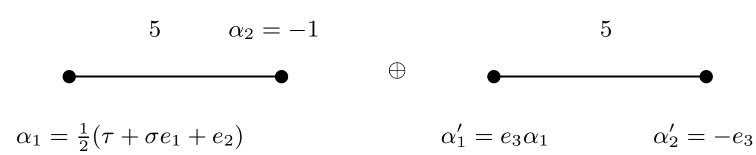

FIG. 3: Coxeter diagram of $H_2 \oplus H'_2$ with quaternions.

by $b = \frac{1}{2}(\tau + \sigma e_1 + e_2)$, then $b^5 = -1$ and the set of roots of $H_2 \oplus H'_2$ will be given by

$$H_2 \ : \ \pm b, \pm b^2, \pm\overline{b}^2, \pm\overline{b}, \pm 1 \tag{16}$$

$$H'_2 \ : \ \pm e_3 b, \pm e_3 b^2, \pm e_3\overline{b}^2, \pm e_3\overline{b}, \pm e_3. \tag{17}$$

We note that (16-17) are the quaternion elements of the dihedral group of order 20 given in (6). They can be more compactly represented as $H_2 : \pm b^m, H_2' : \pm e_3 b^m$ $(m = 1, 2, 3, 4, 5)$. The group $W(H_2)$ is generated by the reflections in the hyperplanes orthogonal to the simple roots α_1 and α_2. $W(H_2)$ can be enumarated explicitly and compactly:

$$W(H_2) = \left\{[b^m, -b^m]^*\right\} \cup \left\{[b^m, b^m]\right\}, \quad m = 1, \ldots, 5. \tag{18}$$

Similarly, we can obtain the group elements of $W(H_2')$ as

$$[e_3 b^m, -e_3 b^m]^*, \quad [b^m, \bar{b}^m], \quad (m = 1, 2, 3, 4, 5). \tag{19}$$

Now each group can be extended to $Aut(H_2)$ and $Aut(H_2')$ by the respective diagram symmetries $\delta_2 : \alpha_1 \leftrightarrow \alpha_2$ and $\delta_2' : \alpha_1' \leftrightarrow \alpha_2'$ where $\delta_2 = [\bar{b}^2, \bar{b}^2]^*$ and $\delta_2' = [e_3 \bar{b}^2, e_3 \bar{b}^2]^*$. Each automorphism group possesses 20 elements and can be represented by the following sets of elements:

$$Aut(H_2) \ : \ [b^m, \pm b^m], [b^m, \pm b^m]^* \tag{20}$$

$$Aut(H_2') \ : \ [e_3 b^m, \pm e_3 b^m]^*, [b^m, \pm \bar{b}^m], \quad m = 1, \ldots, 5. \tag{21}$$

An analysis similar to (13) and (14) can be carried out. We note that the longest elements of $W(H_2)$ and $W(H_2')$ are $w_0 = r_1 r_2 r_1 r_2 r_1 = [-b^3, b^3]^*$, where r_1 and r_2 are the reflection generators of $W(H_2)$ on the roots α_1 and α_2 respectively. A similar consideration leads to the longest element of $W(A_2')$ $w_0' = [-e_3 b^3, e_3 b^3]^*$. We can obtain the products $w_0 \delta = w_0' \delta' = [-1, 1] = c$ which commutes with the elements of $W(H_2)$ and $W(H_2')$. Therefore we can write the groups $Aut(H_2)$ and $Aut(H_2')$ as $Aut(H_2) \approx W(H_2) \times Z_2$ and $Aut(H_2') \approx W(H_2') \times Z_2$. One can also check that $W(H_2)$ and $W(H_2')$ are conjugates under the action of $Z_4 =< [e_3, 1] >$ and thus the maximal group of order 400 has the structure $[W(H_2) \times (H_2')] \rtimes Z_4$. The group elements can be put into the form $[p, q], [p, q]^*$ where p and q take values from the set of roots of $H_2 \oplus H_2'$

$$p, q \in \{\pm b^m, \pm e_3 b^m\}, \quad m = 1, 2, 3, 4, 5. \tag{22}$$

The group generated by p, q pairs consist of 400 elements. The group $Aut[H_2 \oplus H_2'] \approx [W(H_2) \times W(H_2')] \rtimes Z_4$ is maximal as a consequence of the arguments used in Sec. .

THE MAXIMAL SUBGROUP $W(D_4) \rtimes Z_3$ OF ORDER 576

This group has been discussed in reference [12] in a different context. A brief consideration may be in order. The root system of $D_4 \approx SO(8)$ can be generated by the simple roots of quaternions shown in Figure 4.

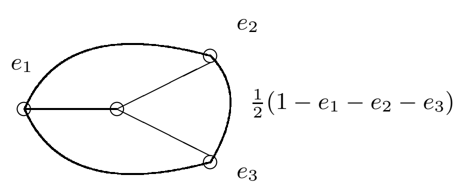

FIG. 4: Dynkin diagram of $SO(8)$ with quaternionic simple roots.

The set of roots constitute the elements of the binary tetrahedral group T, a maximal subgroup of the binary icosahedral group I. The Weyl group $W(D_4)$ of order 192 with 13 conjugacy classes can be generated by the reflection generators $[-e_1, e_1]^*, [-e_2, e_2]^*, [-e_3, e_3]^*$ and $[\frac{1}{2}(-1 + e_1 + e_2 + e_3), \frac{1}{2}(1 - e_1 - e_2 - e_3)^*]$. The automorphism group of the $SO(8)$ root system with the inclusion of Z_3-Dynkin diagram symmetry is generated by $W(D_4)$ along with $[\frac{1}{2}(1 + e_1 + e_2 + e_3), \frac{1}{2}(1 - e_1 - e_2 - e_3)]$, a cyclic symmetry of the diagram for $W(D_4)$. This gives a subgroup of $W(H_4)$ of order 576. We note in passing that the full automorphism group of the root system of $SO(8)$ is $W(D_4) \rtimes S_3$ of order 1152 and it is not a subgroup of $W(H_4)$.

When the quaternions p, q take values from the root system of $SO(8)$

$$p, q \in T = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}\left(\pm 1 \pm e_1 \pm e_2 \pm e_3\right)\} \tag{23}$$

the group consisting of elements $[p,q],[p,q]^*$ forms the desired maximal subgroup of order 576. Under the action of $W(D_4)$ the 120 roots of H_4 split into four sets of elements

$$120 = 24 + 32_1 + 32_2 + 32_3. \tag{24}$$

The first 24 represents the roots of $SO(8)$ and each set of 32 elements is associated with one of the quaternionic units e_i. The Z_3 symmetry of the Dynkin diagram permutes these three sets of 32 elements. One of these three sets of 32 elements can be represented by the quaternions

$$32_1 : \frac{1}{2}(\pm\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(\pm\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(\pm 1 \pm \tau e_1 \pm \sigma e_2), \frac{1}{2}(\pm\sigma e_1 \pm \tau e_2 \pm e_3). \tag{25}$$

The other sets of 32 elements are obtained by cyclic permutations of e_1, e_2 and e_3. However, under the action of the group $W(D_4) \rtimes Z_3$ the 120 elements split as $120 = 24 + 96$. The group $W(D_4) \rtimes Z_3$ is maximal up to conjugacy in the group $W(H_4)$ as argued in Sec. .

THE MAXIMAL SUBGROUP $W(H_3) \times Z_2$

The Coxeter group $W(H_3)$ has many applications in physics. It is the symmetry of an icosahedron with inversion and is isomorphic to the group $A_5 \times Z_2$ of order 120 where A_5 is the group of even permutations of five letters. The C_{60} molecule is the popular example possessing a truncated icosahedral structure. Some metal alloys also display quasicrystallographic aspects with H_3 symmetry. The Coxeter diagram of H_3 can be represented by pure quaternionic simple roots as shown in Figure 5. The roots of H_3 obtained by reflections in the hyperplanes orthogonal to the simple roots of H_3 [13] form a set of 30 pure quaternions which constitute one conjugacy class in I (see Table I). Deleting the right most root $\frac{1}{2}(\sigma + e_2 + \tau e_3)$ of H_4 in Figure 1 we obtain the root diagram of H_3 as shown in Figure 5. The

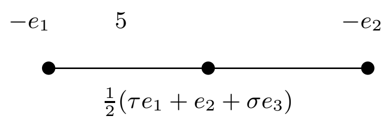

FIG. 5: The Coxeter diagram of H_3 with pure quaternions of I.

generators of $W(H_3)$ $\quad [-e_1, e_1]^*, [-\frac{1}{2}(\tau e_1 + e_2 + \sigma e_3), \frac{1}{2}(\tau e_1 + e_2 + \sigma e_3)]^*$ and $[-e_2, e_2]^*$ generate the group $W(H_3)$ which can be put into the form

$$W(H_3): \ [p, \overline{p}], [p, \overline{p}]^* \tag{26}$$

where p takes any one of the 120 elements of I. In fact $\{[p,\overline{p}] \quad | \quad p \in I\}$ is isomorphic to A_5, a group of order 60 which is the largest finite subgroup of $SO(3)$. The $[p,\overline{p}]^*$ are obtained simply by multiplying $[p,\overline{p}]$ by the conjugation element $[1,1]^*$ which takes $q \to \overline{q}$. The group element $[1,1]^*$ commutes with the elements $[p,\overline{p}]$ implying the structure of $W(H_3) \approx A_5 \times Z_2$. The five conjugacy classes of A_5 consist of the sets of elements

$$[1,1], [12_+, \overline{12}_+], [12'_+, \overline{12'}_+], [20_+, \overline{20}_+], [15_+, \overline{15}_+]. \tag{27}$$

The group $W(H_3)$ with 10 conjugacy classes splits the roots of H_4, as expected, into nine sets of elements as shown in Table I, because $\overline{p} = p^{-1}$ and thus the elements $[p,\overline{p}]$ of $W(H_3)$ act by conjugation on I. It is obvious that the roots of H_3 are left invariant by the element $[-1,1]$ which simply takes any root of H_3 to its negative. But $[-1,1]$ cannot be generated from the H_3 diagram by reflections. Moreover $[-1,1]$ commutes with the elements in (26). Therefore the extended symmetry of the H_3 root system is $W(H_3) \times Z_2$ for $[-1,1]^2 = [1,1]$. The group $W(H_3) \times Z_2$ of order 240 is maximal in $W(H_4)$ and can be written as $A_5 \times Z_2^2$. The element $[-1,1]$ transforms the negative $(-)$ and positive $(+)$ conjugacy classes of I into each other $1 \leftrightarrow -1, 12_+ \leftrightarrow 12_-, 12'_+ \leftrightarrow 12'_-, 20_+ \leftrightarrow 20_-, 30 \leftrightarrow 30$ so that the roots of H_4 decompose into five disjoint sets consisting of elements $2, 24, 24', 40$ and 30. The second Z_2 generator $[-1,1]$ can be taken as a coset representative and the group $W(H_3) \times Z_2$ consisting of 240 elements can be written as

$$[\pm p, \overline{p}], [\pm p, \overline{p}]^*. \tag{28}$$

This is just one way of embedding $W(H_3) \times Z_2$ in $W(H_4)$. Since the index of $W(H_3) \times Z_2$ in $W(H_4)$ is 60 there are 60 different choices for the representations of $H_3 \times Z_2$ in $W(H_4)$.

THE $Aut(A_4)$ AS THE MAXIMAL SUBGROUP OF $W(H_4)$

The automorphism group of the root system of $A_4 \approx SU(5)$ is the semi-direct product of the Weyl group $W(A_4)$ with the Z_2 symmetry of the Dynkin diagram

$$Aut(A_4) \approx W(A_4) \rtimes Z_2. \tag{29}$$

Since the Weyl group $W(A_4)$ is isomorphic to S_5 of order 120 with seven conjugacy classes, the $Aut(A_4)$ is a group of order 240. In what follows, we will study some details of this group.

The Dynkin diagram of $SU(5)$ with quaternionic simple roots can be obtained from Figure 1 by deleting the left-most root $(-e_1)$ and adding another root to the right of the right-most root $\frac{1}{2}(\sigma+e_2+\tau e_3)$. However, for the simplicity of calculations of group elements, we choose the root system of the Dynkin diagram A_4 as shown in Figure 6. The

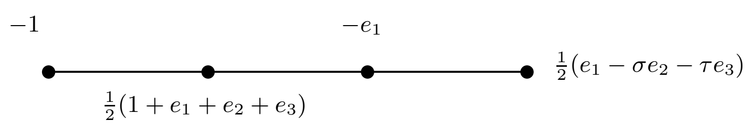

FIG. 6: The Dynkin diagram of $SU(5)$ with quaternion simple roots.

Weyl group $W(A_4)$ is generated by reflections on the simple roots and can be written as

$$\begin{aligned} &[-1, 1]^*, [-\frac{1}{2}(1 + e_1 + e_2 + e_3), \frac{1}{2}(1 + e_1 + e_2 + e_3)]^*, \\ &[-e_1, e_1]^*, [-\frac{1}{2}(e_1 - \sigma e_2 - \tau e_3), \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3)]^*. \end{aligned} \tag{30}$$

These elements generate a group of order 120 isomorphic to the permutation group S_5. The group elements of $W(A_4)$ can be put into the form

$$[p, -a\overline{p}'a], [p, a\overline{p}'a]^*. \tag{31}$$

Here $a = \frac{1}{\sqrt{2}}(e_2 - e_3)$ is a pure quaternion and p', obtained from p by exchanging $\tau \leftrightarrow \sigma$, is an element of the second quaternionic representation I' of the binary icosahedral group obtained from I by exchanging $\tau \leftrightarrow \sigma$ as we mentioned in Section . The transformation $\pm a\overline{p}'a$ converts an element p' of I' back to an element of I. To check the validity of (31) we look into the nontrivial example $[-\frac{1}{2}(e_1 - \sigma e_2 - \tau e_3), \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3)]^*$. Denote by p the element $\frac{1}{2}(e_1 - \sigma e_2 - \tau e_3)$, then p' will be given by $p' = \frac{1}{2}(e_1 - \tau e_2 - \sigma e_3)$ which is an element of the representation I'. Multiplying $\overline{p}'$ by a on the left and right we obtain

$$-a\overline{p}'a = \frac{1}{\sqrt{2}}(e_2 - e_3)\frac{1}{2}(e_1 - \tau e_2 - \sigma e_3)\frac{1}{\sqrt{2}}(e_2 - e_3) = \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3) = p \tag{32}$$

which converts the $[-p, p]^*$ element into the general form $[p, -ap'a]^*$.

To show that the group represented above is closed under multiplication, we verify a non-trivial case. Consider two elements $[p_1, a\overline{p_1}'a]^*$ and $[p_2, -a\overline{p_2}'a]$. In the product $[p_1, a\overline{p_1}'a]^*[p_2, -a\overline{p_2}'a] = [-p_1ap_2a, \overline{p_2}'a\overline{p_1}'a]^*$ if we let $p_3 = -p_1ap_2a$ and compute $a\overline{p_3}'a$ we obtain $\overline{p_2}'a\overline{p_1}'a$, so that the product can be written in the form $[p_3, a\overline{p_3}'a]^*$. Other products can be handled in an anologous fashion. Therefore the set of elements in (31) form a group of order 120. It can be verified that the transformation $a\overline{p}'a$ exchanges the following conjugacy classes of I :

$$\pm 1 \leftrightarrow \pm 1, 12_{\pm} \leftrightarrow 12'_{\pm}, 20_{\pm} \leftrightarrow 20_{\pm}, 15_{\pm} \leftrightarrow 15_{\pm}. \tag{33}$$

The group elements $[p, -a\overline{p}'a]$ form a subgroup of order 60 isomorphic to the group A_5 of even permutations of five letters whose elements can be written in terms of its five conjugacy classes as

$$[1, 1], [15_+, 15_+], [20_+, 20_+], [12_+, 12'_+], [12'_+, 12_+]. \tag{34}$$

It is interesting to note the two different realizations of A_5 in $W(H_4)$ by comparing (34) and (27). We can check that the group element $[-1,1]^*$ which represents reflection with respect to the root 1 leaves the set of elements $[p, -a\overline{p}'a]$ of A_5 invariant under conjugation:

$$[-1,1]^*[p,-a\overline{p}'a][-1,1]^* = [ap'a, -\overline{p}] = [q, -a\overline{q}'a] \tag{35}$$

This proves that the group elements in (31) can be written as a union of two cosets

$$[p,-a\overline{p}'a], [p,-a\overline{p}'a][-1,1]^* \tag{36}$$

implying that the group structure is

$$W(A_4) \approx S_5 \approx A_5 \rtimes Z_2 \tag{37}$$

The roots of H_4 decompose under the Weyl group $W(A_4)$ as sets of 20, 20_+, 20_-, 30_+ and 30_- elements.

The Dynkin diagram symmetry of $SU(5)$ in Figure 6 exchanges the simple roots

$$\begin{array}{lcl} -1 & \leftrightarrow & \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3) \\ -e_1 & \leftrightarrow & \frac{1}{2}(1 + e_1 + e_2 + e_3) \end{array} \tag{38}$$

which can be obtained by the transformation $\gamma = [b, c]$ with $b = \frac{1}{2}(-\tau e_1 + e_2 + \sigma e_3)$ and $c = \frac{1}{2}(\sigma e_1 - \tau e_2 - e_3)$. Then the Weyl group $W(A_4)$ can be extended to the full automorphism group $Aut(A_4)$ of the root system of $SU(5)$ by adjoining the generator $[b, c]$ to $W(A_4)$. The extended group is of order 240 with the structure $Aut(A_4) \approx W(A_4) \rtimes Z_2$ where Z_2 is generated by γ. We repeat the same argument discussed in the other sections where the product of the largest element w_0 and γ leads to $w_0\gamma = [-1,1] = c$. We can then choose $[-1,1]$ as the coset representative rather then $[b,c]$ in which case the element $[-1,1]$ commutes with the set of elements in (31) and transforms all roots of H_4 to their negatives. Extention of $W(A_4)$ either by $[b,c]$ or by $[-1,1]$ leads to the set of elements

$$[p, \mp a\overline{p}'a], [p, \pm a\overline{p}'a]^*. \tag{39}$$

When the elements of $Z_2' =< c = [-1,1] >$ is taken as the coset representatives, the group of order 240 obviously manifests itself as a direct product $W(A_4) \times Z_2'$. We can also prove that $W(A_4) \times Z_2'$ can be embedded in H_4 in 60 different ways.

The maximality of $Aut(A_4)$ is proven with a method analogous to that used at the end of Sec. 3.

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[14] Any unit quaternion $q = -\overline{q}$ is a pure quaternion, where the quaternion conjugate is $\overline{q} = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$
[15] For a detailed study see reference [11]
[16] The reflection of a quaternion q in the plane orthogonal to a root, say a unit quaternion a, can be written as $r_a(q) = q - 2(q,a)a = -a\overline{q}a$.